Equicontinuous factors of S-adic subshifts

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Roscoff workshop: Symbolic dynamics and arithmetic expansions

Joint work with V. Berthé, A. Bustos, P. Cecchi & N. Mañibo

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Motivation

Definition A **minimal topological dynamical systems** (minimal t.d.s) is a tuple (X, T) s.t.

- X is compact and $T\colon X \to X$ is a homeomorphism;
- $\{T^n x : n \in \mathbb{Z}\}$ is dense within X, $\forall x \in X$.

Simplest case: **equicontinuous systems**.

- Let G be a compact abelian group and $a \in G$. Then, $(G, x \mapsto x + a)$ is called an equicontinuous system.

Motivation

- An effective method for studying t.d.s. is through **extensions and factors**.
- A factor between the t.d.s. (X, T) and (Y, S) is a continuous map $\pi\colon X \to Y$ s.t. $\pi \circ T = S \circ \pi$.

Definition. Any minimal t.d.s. (X, T) has a unique **maximal equicontinuous factor** (MEF).

- Any other equicontinuous factor of (X, T) is a factor itself of the MEF.

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The MEF captures important dynamical information.

- Weak mixing: $\exists x, y \in X$ s.t. $\{(T^n x, T^n y) : n \in \mathbb{Z}\}$ is dense in $X \times X$ **iff** the MEF is the trivial one-point system.
- Fibers $\pi^{-1}(y)$ related to nullness, tameness, among others.
- Pisot conjecture states that Pistot substitutional systems are isomorphic to their MEF.

Motivation

- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
- If there exists $\lambda \in \mathbb{T}$ and a factor map $g_\lambda \colon (X, T) \to (\mathbb{T}, \cdot\lambda)$, then λ is called a **continuous eigenvalue**.
- The set of continuous eigenvalues has the structure of a discrete group, and **its Pontryagin dual can be identified with the MEF**.

Motivation

- For any t.d.s. (X, T) there is a (typically non-unique) ergodic measure μ.
- This gives rise of an ergodic measure-preserving system (X, T, μ).
- Analogous notions for equicontinuous systems, factors, eigenvalues and MEF (which in this case are called **measurable eigenvalues** and the **Kronecker factor**!)

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- Analogous notions for equicontinuous systems, factors, eigenvalues and MEF (which in this case are called **measurable eigenvalues** and the **Kronecker factor**!)
- Intrincate interplay between topological and measure-theoretic notions.
- **Objective**: Study the MEF (Kronecker) factors of minimal (ergodic) subshifts.

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1. Equicontinuous factors
2. The Maximal Equicontinuous Factor
3. The Kronecker factor

Substitutions

- Early examples of minimal systems: **substitutional subshifts**.

Subshifts: (X, S), with S the **shift** and $X \subseteq \mathcal{A}^{\mathbb{Z}}$ closed and S-invariant.

- Its **language** $\mathcal{L}(X) = \{x_i x_{i+1} \dots x_{j-1} : x \in X, i < j\}$.

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Substitutions.

- Endomorphisms $\tau \colon \mathcal{A}^* \to \mathcal{A}^*$ s.t. $\tau(a) \neq 1, \forall a \in \mathcal{A}$.
- We assume **primitivity**: $\exists N > 0$, $\forall a, b \in \mathcal{A}$, a appears in $\tau^N(b)$.
- $\mathcal{L} = \{u : u \text{ occurs in } \tau^n(a) \text{ for some } n > 0, a \in \mathcal{A}\}$.
- Let $X_\tau = \{x \in \mathcal{A}^{\mathbb{Z}} : \forall k, x_{-k} x_{-k+1} \dots x_k \in \mathcal{L}\}$.
- X_τ is minimal and has a unique ergodic measure μ.

Substitutions

Consider $\mathcal{A} = \{0, 1, 2\}$ and

$$\tau\colon \begin{cases} 0 & \mapsto 010 \\ 1 & \mapsto 102 \\ 2 & \mapsto 201 \end{cases}$$

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$x = \ldots 010\,102\,010\,102\,0.10\,201\,010\,102\,010 \ldots \in X_\tau$

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					0					
		0			1			0		
$y =$ ...	0	1	0	1	.0	2	0	1	0	... $\in X_\tau$
$x =$ ...	010	102	010	102	01.0	201	010	102	010	... $\in X_\tau$

$$x = S^2\tau(y).$$

Early results: F.M. Dekking*

- τ of **constant-length**: $|\tau| := |\tau(a)| = |\tau(b)|, \forall a, b \in \mathcal{A}$.
- Example. $\tau : 0 \mapsto 010,\ 1 \mapsto 102,\ 2 \mapsto 201$.

*Dekking, F.M. The spectrum of dynamical systems arising from substitutions of constant length. 1978

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- Example. $\tau : 0 \mapsto 010,\ 1 \mapsto 102,\ 2 \mapsto 201$.

Theorem. There exists $h(\tau) \in \mathbb{N}$ (the **height**) s.t.

$$\mathrm{Kro}(X_\tau) = \mathbb{Z}_{|\tau|} \times \mathbb{Z}/h(\tau)\mathbb{Z}, \quad (\mathbb{Z}_{|\tau|} = |\tau|\text{-adic integers})$$

Moreover, $\exists c(\tau) \in \mathbb{N}$ (the **column number**) s.t.

$$\pi_{\mathrm{Kro}} : X_\tau \to \mathrm{Kro}(X_\tau)$$

satisfies $\mathrm{Haar}(\{y \in \mathrm{Kro}(X_\tau) : \#\pi_{\mathrm{Kro}}^{-1}(y) = c(\tau)\}) = 1$.

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Variable length: B. Host[†]

- The MEF and Kronecker factors coincide.
- A (letter-) **coboundary** in X_τ is a morphism $c\colon \mathcal{A}^* \to \mathbb{T}$ s.t. $\forall a \in \mathcal{A}, \forall aua \in \mathcal{L}(X_\tau) : c(au) = 1$.
- Let $\ell_n \in \mathbb{Z}^{\mathcal{A}}$ defined by $\ell_n(a) = |\tau^n(a)|$.

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Theorem. $\lambda \in \mathbb{T}$ is an eigenvalue of X_τ **iff** there is $p > 0$ and a coboundary c in X_τ s.t.

$$c(a) = \lim_{n\to\infty} \lambda^{\ell_{pn}(a)}, \quad \forall a \in \mathcal{A}.$$

The fibers are much more difficult to understand.

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Let $\mathcal{A} = \{0, 1, 2, 3, 4\}$ and

$$\tau : \begin{cases} 0 & \mapsto 34 \\ 1 & \mapsto 01 \\ 2 & \mapsto 234 \\ 3 & \mapsto 34 \\ 4 & \mapsto 012 \end{cases}$$

- Then $\exp(2\pi i\sqrt{2})$ is an eigenvalue value defining the coboundary $c(a) = 1$ if $a \neq 1$, $c(1) = 0.707\ldots$.

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- In fact, X_τ is (conjugate to) the Sturmian of angle $\sqrt{2}$.

Beyond substitutions: S-adic formalism

- Substitutions $\tau \colon \mathcal{A}^* \to \mathcal{B}^*$ between **different** alphabets.
- A **directive sequence** is a sequence of substitutions $\boldsymbol{\tau} = (\tau_n : n \geq 0)$ of the form

$$\mathcal{A}_0^* \xleftarrow{\tau_0} \mathcal{A}_1^* \xleftarrow{\tau_1} \mathcal{A}_2^* \xleftarrow{\tau_2} \mathcal{A}_3^* \xleftarrow{\tau_3} \mathcal{A}_4^* \xleftarrow{\tau_4} \dots$$

  - Abbreviate $\tau_{n,N} = \tau_n \circ \tau_{n+1} \circ \cdots \circ \tau_{N-1}$.
- Assume that $\boldsymbol{\tau}$ is **primitive**: $\forall n \geq 0, \exists N > n$ s.t.

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Definition. $\boldsymbol{\tau}$ generates the **S-adic subshift**:

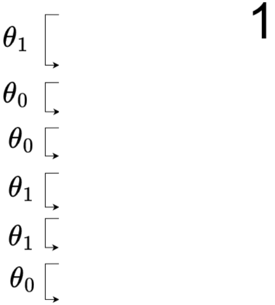
$$X_{\boldsymbol{\tau}} = \{x \in \mathcal{A}_0^{\mathbb{Z}} : \forall k, x_{[-k,k]} \text{ appears in } \tau_{0,n}(a) \text{ for some } n > 0, a \in \mathcal{A}_n\}.$$

$X_{\boldsymbol{\tau}}^{(n)} = X_{S^n \boldsymbol{\tau}}$, with $S^n \boldsymbol{\tau} = (\tau_n, \tau_{n+1}, \dots)$.

S-adic formalism: Sturmians

Theorem. X is Sturmian iff $X = X_{\boldsymbol{\tau}}$ for some $\boldsymbol{\tau} = (\tau_n : n \geq 0)$ where each τ_n is one of the following substitutions:

$$\theta_0 : \begin{cases} 0 & \mapsto 01 \\ 1 & \mapsto 0 \end{cases} \qquad \theta_1 : \begin{cases} 0 & \mapsto 1 \\ 1 & \mapsto 10 \end{cases}$$



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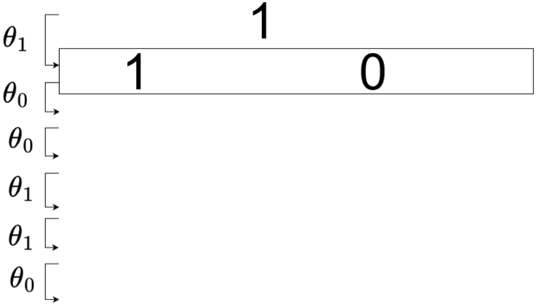

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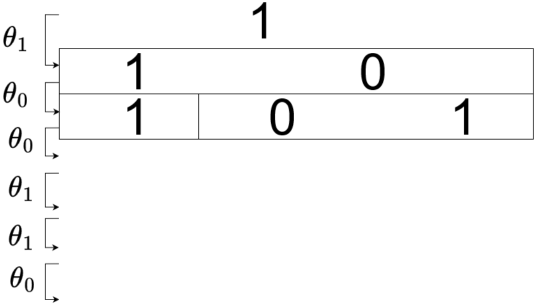

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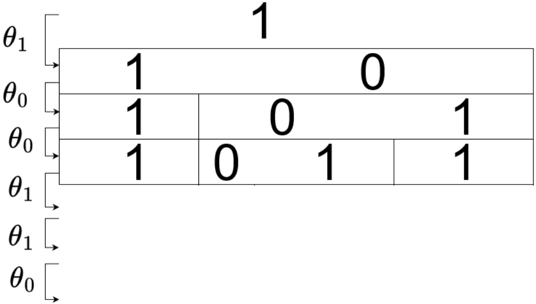

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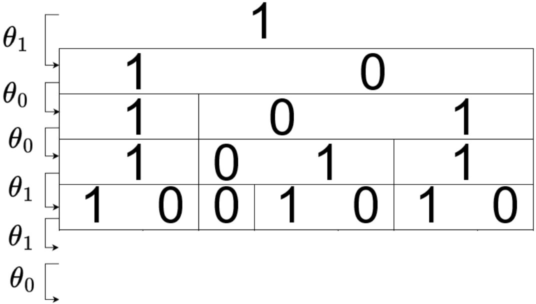

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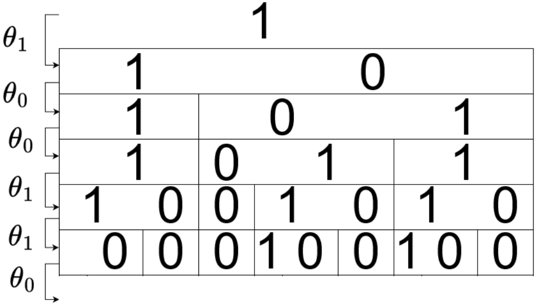

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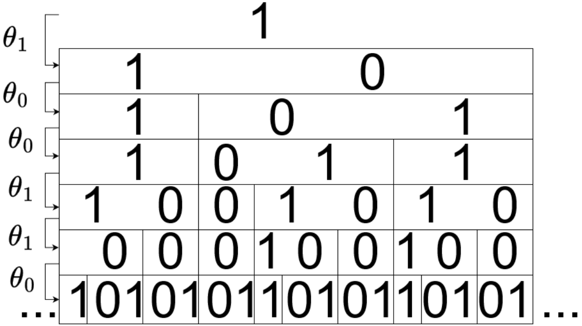

Beyond substitutions: S-adic formalism

- We always consider **recognizable** directive sequences.
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Beyond substitutions: S-adic formalism

- We always consider **recognizable** directive sequences.
- $X_{\boldsymbol{\tau}}$ is minimal, but it might have many ergodic measures.
- **Any** minimal subshift X is equal to $X_{\boldsymbol{\tau}}$ for some $\boldsymbol{\tau}$.
- A **contraction** has the form $\boldsymbol{\tau}' = (\tau_{n_k,n_{k+1}})_{k\geq 0}$ for $0 = n_0 < n_1 < \dots$

Linearly recurrent subshifts (LR)

- X is linearly recurrent **iff** $X = X_{\boldsymbol{\tau}}$ for some $\boldsymbol{\tau}$ s.t.
  - **finitary**: there are finitely many different τ_n.
  - $\boldsymbol{\tau}$ is **strongly primitive**: a occurs in $\tau_n(b)$, $\forall n, a, b$.
  - the τ_n are **proper**: $\exists a \in \mathcal{A}_n$ s.t. $\tau_n(b)$ starts with a, $\forall b$.

- $\tau\colon 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0.$

- In Sturmians: substitutions $\leftrightarrow$ quadratic irrationals;
  LR $\leftrightarrow$ bounded coefficients in the continued fraction.

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- In Sturmians: substitutions $\leftrightarrow$ quadratic irrationals;
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Theorem. $\lambda \in \mathbb{T}$ is a continuous eigenvalue **iff**[a]

$$\sum_{n \geq 0} \max_{a \in \mathcal{A}_n} |\lambda^{|\tau_{0,n}(a)|} - 1| < \infty.$$

[a] Cortéz, Durand, Host, Maass. Continuous and measurable eigenfunctions of linearly recurrent dynamical Cantor systems. 2003.

Linearly recurrent subshifts (LR)

- LR systems have a unique ergodic measure.

Theorem. $\lambda \in \mathbb{T}$ is a measurable eigenvalue **iff**

$$\sum_{n\geq 0} \max_{a\in\mathcal{A}_n} |\lambda^{|\tau_{0,n}(a)|} - 1|^2 < \infty.$$

Continuous and measurable eigenvalues might differ!

Topological developments

- Let $\boldsymbol{\tau}$ be a **strongly primitive** and **proper directive** sequence.
- Let $\mathrm{pref}(u)$ be the set of prefixes of $u \in \mathcal{A}^*$; $\ell_n(u) = |\tau_{0,n}(u)|$.

Theorem. $\lambda \in \mathbb{T}$ is a continuous eigenvalue **iff**[a][b]

$$\sum_{n \geq 0} \max_{a \in \mathcal{A}_{n+1}} \max_{ua \in \mathrm{pref}(\tau_n(a))} |\lambda^{\ell_n(u)} - 1| < \infty.$$

[a] Bressaud, Durand, Maass. Eigenvalues of finite rank Bratteli-Vershik systems. 2012.

[b] Durand, Frank, Maass. Eigenvalues of minimal Cantor systems. 2015.

The non-proper case

- Interval Exchange Transformations:
  - Their symbolic codings have nice S-adic expansions.
  - S-adic encodes the Rauzy induction.
  - Access to Theichmuller related machinery.

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- Linear involutions.
  - Symbolic codings have non-proper S-adic expansions.
  - S-adic encodes the Rauzy induction.
  - No access to Theichmuller machinery due to symbolic techniques relying on properness.

Coboundaries

- A morphism $c\colon \mathcal{A}^* \to \mathbb{T}$ is a **coboundary** in X if $c(au) = 1, \forall aua \in \mathcal{L}(X)$.
- There exists $\rho\colon \mathcal{A} \to \mathbb{T}$ s.t. $\rho(a)c(a) = \rho(b)$, $\forall ab \in \mathcal{L}(X)$.

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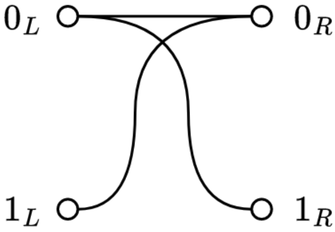
- Let $\mathcal{E}(X)$ be graph with vertices $\{a_L, a_R : a \in \mathcal{A}\}$ and edges (a_L, b_R) for $ab \in \mathcal{L}(X)$.
- Ex. Let X be the Sturmian of angle $\frac{\sqrt{5}-1}{2}$. So, $x = \ldots 010010 \cdots \in X$ and $\mathcal{L}(X) \cap \mathcal{A}^2 = \{00, 01, 10\}$.



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- Let $\mathcal{E}(X)$ be graph with vertices $\{a_L, a_R : a \in \mathcal{A}\}$ and edges (a_L, b_R) for $ab \in \mathcal{L}(X)$.
- Conversely, if $\rho\colon \mathcal{A} \to \mathbb{T}$ satisfies $\rho(a) = \rho(a')$ for all a_R, a'_R in the same connected component of $\mathcal{E}(X)$, then $c(a) := \rho(b)\rho(a)^{-1}$ defines a coboundary in X.

The non-proper case

- Let $\boldsymbol{\tau}$ be **finitary** (there are finitely many τ_n), *straight* and with *fully essential words*.

Theorem (Berthé, Cecchi,Yassawi 2022). If λ is a continuous eigenvalue, then

$$c(a) = \lim_{n\to\infty} \lambda^{\ell_n(a)} \text{ defines a } \textit{weak S-adic coboundary}.$$

- A sufficient condition is provided as well.

- Similar results by P. Mercat (April 2024).

The non-proper case

- Let $\boldsymbol{\tau}$ be **primitive**.

Theorem (Berthé, Cecchi, E.) λ is a continuous eigenvalue of $X_{\boldsymbol{\tau}}$ **iff** there exists $(\rho_n(a) \in \mathbb{T} : a \in \mathcal{A}_n)_{n \geq 0}$ s.t.

$$\max_{a_k \in \mathcal{A}_k} \max_{u_k a_k \in \mathrm{pref}(\tau_k(a_{k+1}))} \left| \lambda^{\sum_{n \leq k < N} \ell_k(u_k)} - \rho_N(a_N) \right| \qquad (\triangle)$$

converges to 0 as $n, N \to \infty$, *i.e.*,

$$\forall \epsilon > 0,\ \forall n \gg_\epsilon 1 : (\triangle) < \epsilon \text{ for all large enough } N > n.$$

Simpler expressions if $\boldsymbol{\tau}$ is strongly primitive/finitary/proper.
The $\rho_n(a)$ do **not** always define coboundaries.

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$$\sum_{n\leq k<N} \max_{a_k\in\mathcal{A}_k} \max_{u_k a_k \in \mathrm{pref}(\tau_k(a_{k+1}))} \left|\lambda^{\ell_k(u_k)} - \rho_k(a_k)\right| \qquad (\triangle)$$

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The $\rho_n(a)$ do **not** always define coboundaries.

The non-proper case

- Let $\boldsymbol{\tau}$ be **primitive** and **proper**.

Theorem (Berthé, Cecchi, E.) λ is a continuous eigenvalue of $X_{\boldsymbol{\tau}}$ **iff**

$$\max_{a_k \in \mathcal{A}_k} \max_{u_k a_k \in \mathrm{pref}(\tau_k(a_{k+1}))} \left|\lambda^{\sum_{n \le k < N} \ell_k(u_k)} - 1\right| \qquad (\triangle)$$

converges to 0 as $n, N \to \infty$, *i.e.*,

$$\forall \epsilon > 0,\ \forall n \gg_\epsilon 1 : (\triangle) < \epsilon \text{ for all large enough } N > n.$$

Simpler expressions if $\boldsymbol{\tau}$ is strongly primitive/finitary/proper.
The $\rho_n(a)$ do **not** always define coboundaries.

Finite alphabet rank

- Assume that $\sup_{n\to\infty} \#\mathcal{A}_n < \infty$ (**finite alphabet rank**).
- Then, the $\rho_n(a)$ define coboundaries in $X_{\boldsymbol{\tau}}^{(n)}$.

Theorem (Berthé, Cecchi, E.) λ is a continuous eigenvalue of $X_{\boldsymbol{\tau}}$ **iff** there exist coboundaries c_n in $X_{\boldsymbol{\tau}}^{(n)}$ s.t.

$$\max_{a_k \in \mathcal{A}_k} \max_{u_k a_k \in \mathrm{pref}(\tau_k(a_{k+1}))} \left|\lambda^{\sum_{n\leq k<N} \ell_k(u_k)} - \prod_{n\leq k<N} c_k(u_k)\right| \qquad (\triangle)$$

converges to 0 as $n, N \to \infty$, *i.e.*,

$$\forall \epsilon > 0,\ \forall n \gg_\epsilon 1 : (\triangle) < \epsilon \text{ for all large enough } N > n.$$

Similar simpler expressions if $\boldsymbol{\tau}$ is strongly primitive/finitary/proper.

Finite alphabet rank

- We are led to study $(\mathcal{E}(X_\tau^{(n)}) : n \geq 0)$.
- Any $\rho\colon \mathcal{A} \to \mathbb{T}$ defining a cob. in X takes at most $\mathrm{cc}(\mathcal{E}(X))$ different values.

[‡] Already in Berthé, Yassawi & Cecchi's paper

Finite alphabet rank

- We are led to study $(\mathcal{E}(X_{\boldsymbol{\tau}}^{(n)}) : n \geq 0)$.
- Any $\rho \colon \mathcal{A} \to \mathbb{T}$ defining a cob. in X takes at most $\mathrm{cc}(\mathcal{E}(X))$ different values.
- Examples:
  - IETs: $\mathcal{E}(X_{\boldsymbol{\tau}}^{(n)})$ is a tree; hence, all coboundaries are trivial $c_n \equiv 1$.
  - Linear involutions: $\mathcal{E}(X_{\boldsymbol{\tau}}^{(n)})$ has two connected components; hence, $\#\{\rho_n(a) : a \in \mathcal{A}_n\} = 2$.
  - Brun substitutions: For $i, j \in \{0, 1, \ldots, d-1\}$, $i \neq j$:

$$\sigma_{i,j} \colon \begin{cases} j & \mapsto ij \\ k & \mapsto k \text{ if } k \neq j \end{cases}$$

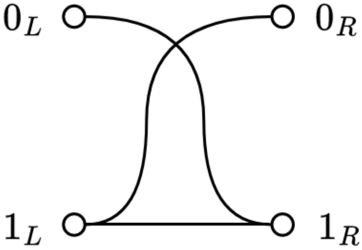
    Then, $\mathcal{E}(X_{\boldsymbol{\tau}}^{(n)})$ is connected; hence, $c_n \equiv 1$ for all coboundaries[‡].

[‡] Already in Berthé, Yassawi & Cecchi's paper

Further examples

- Let $\mathcal{E}(X, u)$ be the graph with edges $\{(a_L, b_R) : aub \in \mathcal{L}(X)\}$.
- Let $x = \ldots 01001010 \cdots \in X_{\mathrm{Fibonnaci}}$.

$$\mathcal{E}(X, 0)$$



Further examples

- Let $\mathcal{E}(X, u)$ be the graph with vertex set $\{a_L, a_R : a \in \mathcal{A}\}$ and edge set $\{(a_L, b_R) : aub \in \mathcal{L}(X)\}$.

- X is **dendric** if $\forall u \in \mathcal{L}(X) : \mathcal{E}(X, u)$ is a tree.

Further examples

- Let $\mathcal{E}(X, u)$ be the graph with vertex set $\{a_L, a_R : a \in \mathcal{A}\}$ and edge set $\{(a_L, b_R) : aub \in \mathcal{L}(X)\}$.
- X is **dendric** if $\forall u \in \mathcal{L}(X) : \mathcal{E}(X, u)$ is a tree.
- Example of dendric subshifts: IETs, Arnoux-Rauzy, systems generated by Cassaigne algorithm, among others.
- Several interesting properties about return words, bifix decoding, dimension group, and others.

Further examples

- Let $\mathcal{E}(X,u)$ be the graph with vertex set $\{a_L, a_R : a \in \mathcal{A}\}$ and edge set $\{(a_L, b_R) : aub \in \mathcal{L}(X)\}$.
- X is **dendric** if $\forall u \in \mathcal{L}(X) : \mathcal{E}(X,u)$ is a tree
- **Prop.**[§] $p_X(n) := \#(\mathcal{L}(X) \cap \mathcal{A}^n) = (\#\mathcal{A} - 1)n + 1$.

Theorem. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ with an ergodic measure μ s.t. the entries of $(\mu([a]) : a \in \mathcal{A})$ are rationally independent. Then:

- $p_X(n) \geq (\#\mathcal{A} - 1)n + 1$.
- $p_X(n) = (\#\mathcal{A} - 1)n + 1, \forall n$, implies that X is dendric.

[§] Berthé *et al.* Acyclic, connected and tree sets. (2015)

Example

- Let $\mathcal{E}(X,u)$ be the graph with edges $\{(a_L, b_R) : aub \in \mathcal{L}(X)\}$.
- X is **dendric** if $\mathcal{E}(X,u)$ is a tree for all $u \in \mathcal{L}(X)$.

For $a \neq b$, let

$$\theta_{a,b} \colon \begin{cases} a & \mapsto ab \\ c & \mapsto c \text{ if } c \neq a \end{cases}$$

(E., Leroy) If X is dendric, then $X = X_{\boldsymbol{\tau}}$ where $\boldsymbol{\tau} = (\tau_n)_{n \geq 0}$ satisfies:

- each τ_n is one of the $\theta_{a,b}$.
- $X_{\boldsymbol{\tau}}^{(n)}$ is dendric for all $n \geq 0$.

Therefore, eigenvalues come from trivial coboundaries $c_n \equiv 1$.

Index

1. Equicontinuous factors
2. The Maximal Equicontinuous Factor
3. The Kronecker factor

The Kronecker factor

- Constant-length substitutions: identification of Kronecker factor and its fibers sizes (Dekking '70s).

- General substitutive case: Kronecker factor is identified; fiber sizes are much more difficult to understand (Host '80s).

- S-adic with finite alphabet rank: Kronecker factor is identified (Durand *et al.* 2019)
  - Properness becomes irrelevant.

The Kronecker factor

- Our focus: S-adic with constant-length; no finite alphabet rank.

- **Any** equicontinuous system is the Kronecker factor of a constant-length S-adic subshift (Williams '84).

- Our focus: Identification of the Kronecker factor and its generic fiber structure.

Theorem (Bustos, Mañibo, E.).

- Formula for the rational part of the Kronecker factor.
- Finite alphabet rank: formula for the generic fiber size.

Thank you!